

On infinite-dimensional limit of the Steinberg representations

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We present a construction of the Steinberg representation admitting an automatic pass to an infinite-dimensional limit.

Recall, that the representation theory of infinite-dimensional classical groups and infinite symmetric groups is a relatively old well-developed topic. For infinite-dimensional groups over finite fields a progress appeared comparatively recently, in [9], [10], [3] and in [6], [7]. This note contains a construction intermediate between these works.

1. Notation. Denote by \mathbb{F}_q the field with q elements, let \mathbb{F}_q^n be the coordinate n -dimensional linear space with the standard basis e_j . Denote by $\mathrm{GL}(n)$ the group of all invertible matrices of order n . It acts on \mathbb{F}_q^n by multiplication $x \mapsto xg$ of a row $x \in \mathbb{F}_q^n$ by a matrix $g \in \mathrm{GL}(n)$.

Denote by S_n the symmetric group. It is generated by the transpositions $\tau_j = (j, j+1)$, the relations are $\tau_j^2 = 1$, $(\tau_j \tau_{j+1})^3 = 1$, and $\tau_k \tau_j = \tau_j \tau_k$ for $|k - j| \geq 2$.

By $\ell_2(Z)$ we denote the space of complex functions on a finite set Z equipped with the ℓ_2 -inner product.

2. Schubert cells. Denote by $\mathrm{Fl}(n)$ the space of complete flags \mathcal{V} in \mathbb{F}_q^n ,

$$\mathcal{V} : 0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{F}_q^n, \quad \dim V_j = j.$$

Fix a standard flag

$$\mathcal{E} : E_0 \subset E_1 \subset \cdots \subset E_n, \quad E_j = \sum_{i \leq j} \mathbb{F}_q e_i.$$

Denote by $B(n) \subset \mathrm{GL}(n)$ the stabilizer of \mathcal{E} . It consists of lower-triangular matrices. For each $\sigma \in S_n$ we consider the flag

$$\mathcal{E}^\sigma : E_0^\sigma \subset E_1^\sigma \subset \cdots \subset E_n^\sigma, \quad \text{where} \quad E_j^\sigma = \mathbb{F}_q e_{\sigma(1)} \oplus \cdots \oplus \mathbb{F}_q e_{\sigma(j)}.$$

Orbits of $B(n)$ on $\mathrm{Fl}(n)$ are called *Schubert cells*, for details, see [2], 10.2. They are enumerated by elements $\sigma \in S_n$: each orbit contains a unique flag \mathcal{E}^σ , we denote such orbit by X^σ .

3. The Steinberg representation. Denote by $\mathrm{Fl}_j(n)$ the space of incomplete flags containing subspaces of all dimensions except j . Denote by $\pi_j : \mathrm{Fl}(n) \rightarrow \mathrm{Fl}_j(n)$ the map forgetting V_j . There is a natural map $\Pi_j : \ell_2(\mathrm{Fl}(n)) \rightarrow \ell_2(\mathrm{Fl}_j(n))$ defined by

$$\Pi_j f(W) = \frac{1}{q+1} \sum_{\mathcal{V} : \pi_j(\mathcal{V}) = W} f(W)$$

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In fact, the summation is taken over all flags

$$W_0 \subset \cdots \subset W_{j-1} \subset Y \subset W_{j+1} \subset \cdots \subset W_n,$$

such flags are enumerated by subspaces Y satisfying $W_{j-1} \subset Y \subset W_{j+1}$, or equivalently over set of lines in the 2-dimensional space W_{j+1}/W_{j-1} .

Theorem 1 *There exists a unique irreducible representation of $\mathrm{GL}(n)$, which is contained in $\ell_2(\mathrm{Fl}(n))$ and does not contained in spaces $\ell_2(\mathrm{Fl}_j(n))$.*

This representation is called a *Steinberg representation*, for its fascinating properties, see surveys [4], [8].

4. Definition of the Steinberg representation via reproducing kernels. We define a function $k(\mathcal{V}, \mathcal{W})$ on $\mathrm{Fl}(n) \times \mathrm{Fl}(n)$ as the number of all pairs (i, j) , where i, j range in $\{0, 1, \dots, n-1\}$, such that

$$\dim V_i \cap W_j = \dim V_{i+1} \cap W_{j+1}. \quad (1)$$

Define the kernel $K(\cdot, \cdot)$ on $\mathrm{Fl}(n)$ by

$$K(\mathcal{V}, \mathcal{W}) = (-q)^{-k(\mathcal{V}, \mathcal{W})}.$$

By definition, the kernel $K(\cdot, \cdot)$ is $\mathrm{GL}(n)$ -invariant.

Proposition 2 *The function*

$$\varkappa(\mathcal{W}) := k(\mathcal{E}, \mathcal{W})$$

is constant on Schubert cells X^σ . The value of \varkappa on X^σ coincides with the number $I(\sigma)$ of inversions of σ . The number of points of X^σ coincides with $q^{k(\mathcal{E}, \mathcal{W})}$.

Notice that $I(\sigma)$ coincides with the length of a shortest decomposition of σ in product of the generators τ_j .

PROOF. Let us evaluate the number of points of X^σ . Consider an example. Let $n = 6$,

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \quad (2)$$

Vectors e_σ are rows of this matrix. A $B(n)$ -orbit of the collection $\{e_\sigma\}$ consists of arbitrary collections of the type

$$\begin{pmatrix} * & * & * & \circ & 0 & 0 \end{pmatrix}, \\ \begin{pmatrix} * & \circ & 0 & 0 & 0 & 0 \end{pmatrix}, \\ \begin{pmatrix} * & * & * & * & \circ & 0 \end{pmatrix}, \\ \begin{pmatrix} \circ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ \begin{pmatrix} * & * & * & * & * & \circ \end{pmatrix}, \\ \begin{pmatrix} * & * & \circ & 0 & 0 & 0 \end{pmatrix},$$

where $*$ denotes arbitrary elements of \mathbb{F}_q and \circ nonzero elements. We have \circ 's on the former positions of units and $*$'s on positions to the left of units. Elements of flags (subspaces) are linear combinations $\sum_{j \leq k} c_j e_{\sigma(j)}$. Replacements

$$e_{\sigma(j)} \rightarrow \lambda e_{\sigma(j)} + \sum_{i < j} a_i e_{\sigma(i)}, \quad \lambda \neq 0,$$

do not change the flag. Therefore we can get 1 on the place of \circ 's and 0 under all units. Thus we get that any flag in $B(n)$ -orbit of \mathcal{E}^σ is generated by a collection of vectors

$$\begin{pmatrix} * & * & * & 1 & 0 & 0 & 0 \end{pmatrix}, \\ \begin{pmatrix} * & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ \begin{pmatrix} * & 0 & * & 0 & 1 & 0 & 0 \end{pmatrix}, \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & 0 & * & 0 & 0 & 1 & 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3)$$

Now for each star we have a unit under this star and a unit to the right of the star. This pair of units corresponds to an inversion in σ .

Next, let us evaluate the number (i, j) of pairs satisfying (1). Dimension of $E_i \cap F_j$ is the number of units in the left upper $i \times j$ corner of the matrix σ . The condition (1) means that $i \times j$ and $(i+1) \times (j+1)$ -corners contain the same units. Therefore units in $(i+1)$ -th row and $(j+1)$ -th column are outside the $(i+1) \times (j+1)$ -corner. Hence we have $*$ on the $(i+1)(j+1)$ -th place in (3). \square

Lemma 3 *The kernel $K(\cdot, \cdot)$ is positive definite².*

Consider the Euclidean space H_n determined by the reproducing kernel³ $K(\mathcal{V}, \mathcal{W})$.

Lemma 4 *The representation of $GL(n)$ in H_n coincides with the Steinberg representation.*

PROOFS OF LEMMAS. By the Frobenius reciprocity, any subrepresentation in $\ell^2(\text{Fl}(n))$ contains a $B(n)$ -invariant vector. Denote by η a $B(n)$ -invariant function in the Steinberg subrepresentation St in ℓ_2 . Denote by $\eta[\sigma]$ its value on a Schubert cell X^σ . By the definition η satisfies the equations $\Pi_j \eta = 0$. It is easy to see that that these equations have the form

$$q\eta[\tau_j \sigma] + \eta[\sigma] = 0 \quad \text{if } I(\tau_j \sigma) > I(\sigma).$$

These recurrence relations have a unique (up to a constant factor) solution, namely $\eta[\sigma] = (-q)^{-I(\sigma)}$. This also proves Theorem 1.

Denote by $M(\cdot, \cdot)$ the reproducing kernel determining the subspace St . This means that the functions

$$\delta_{\mathcal{V}}(\mathcal{W}) = M(\mathcal{V}, \mathcal{W})$$

²I.e., for any collection of points $\mathcal{V}_i \in \text{Fl}(n)$, we have $\det_{i,j} \{K(\mathcal{V}_i, \mathcal{V}_j)\} \geq 0$.

³See, e.g., [5], Section 7.1.

are contained in St and for any function f on $\text{Fl}(n)$ we have

$$f(\mathcal{V}) = \langle f, \delta_{\mathcal{V}} \rangle_{\ell_2(\text{Fl}(n))}.$$

Since St is $\text{GL}(n)$ -invariant, the kernel M is $\text{GL}(n)$ -invariant, $M(g\mathcal{V}, g\mathcal{W}) = M(\mathcal{V}, \mathcal{W})$. Since the action of $\text{GL}(n)$ on $\text{Fl}(n)$ is transitive, the kernel is determined by its values for $\mathcal{V} = \mathcal{E}$, i.e. by the function $\delta_{\mathcal{E}}$. Moreover, $\delta_{\mathcal{E}}(\mathcal{W})$ is $\text{B}(n)$ -invariant, and therefore $\delta_{\mathcal{E}}(\mathcal{W}) = s \cdot \eta(\mathcal{W})$. \square

REMARK. This construction of the Steinberg representation is a rephrasing of [1], Theorem 10.2.

5. Infinite-dimensional limit. Preliminaries. Consider the linear space L , whose vectors are two-side sequences

$$x = (\dots, x_{-1}, x_0, x_1, \dots).$$

such that $x_k = 0$ for sufficiently large k . We represent operators in L as infinite matrices $g = g_{ij}$, where $-\infty < i, j < \infty$. Denote by $\text{B}(2\infty)$ the group of all infinite matrices g such that $g_{ij} = 0$ for $i < j$ and g_{ii} are invertible (i.e., we consider all invertible lower-triangular matrices). Denote by $\text{GL}(2\infty)$ the group of *finitary*⁴ invertible matrices. Denote by $\text{GLB}(2\infty)$ the group of matrices generated by $\text{GL}(2\infty)$ and $\text{B}(2\infty)$. The group $\text{GLB}(2\infty)$ acts in L by transformations $x \mapsto xg$.

Denote by E_j , where $-\infty < j < \infty$, the subspace in L consisting of vectors

$$(\dots, x_{j-1}, x_j, 0, 0, \dots).$$

Thus we get the *standard flag* \mathcal{E} :

$$\dots \subset E_{-1} \subset E_0 \subset E_1 \subset \dots$$

We define a flag space $\text{Fl}(2\infty)$ as the space of complete flags coinciding with the standard flag in all but a finite number of terms. More precisely, consider flags \mathcal{V} having the following form. Fix $M \leq N$. Set $V_j = E_j$ if $j \leq M$ and $j \geq N$. Consider the finite-dimensional space E_N/E_M and a complete flag in E_N/E_M ,

$$0 = F_0 \subset F_1 \subset \dots \subset F_{N-M} = E_N/E_M.$$

For $0 \leq \alpha \leq N-M$, we set $V_{M+\alpha}$ being the preimage of F_α under the projection $E_N \rightarrow E_N/E_M$.

Setting $M = -n$, $N = n$, we get that the space $\text{Fl}(2\infty)$ is an inductive limit of the chain

$$\dots \longrightarrow \text{Fl}(2n+1) \longrightarrow \text{Fl}(2n+3) \longrightarrow \dots$$

The group $\text{GLB}(2\infty)$ acts on the space $\text{Fl}(2\infty)$.

6. The infinite-dimensional limit of Steinberg representations. We define the function $K(\mathcal{V}, \mathcal{W})$ on $\text{Fl}(2\infty) \times \text{Fl}(2\infty)$ as the number of pairs $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ such that

$$V_{i+1} \cap W_{j+1} = V_i \cap W_j.$$

⁴This means that $g - 1$ has finite number of nonzero matrix elements.

The kernel $K(\mathcal{V}, \mathcal{W}) = (-p)^{-k(\mathcal{V}, \mathcal{W})}$ is positive definite on each space $\text{Fl}(2n+1)$ and therefore it is positive-definite on the inductive limit $\text{Fl}(2\infty)$. We consider the Hilbert space determined by the reproducing kernel K and the unitary representation of $\text{GL}(2\infty)$ in this space.

7. Comparison with earlier papers. a) Consider the space L_+ consisting of sequences (x_0, x_1, \dots) such that $x_j = 0$ for all but a finite number of j . The same construction gives a Steinberg representation obtained in [3].

b) Grassmannians and flags in the space L were considered in [6]. However, the topic of [6] is the group of all continuous transformations of (locally compact Abelian group) L ; this group is larger than $\text{GLB}(2\infty)$. Also, [6] treats another space of flags, which have empty intersection with $\text{Fl}(2\infty)$.

References

- [1] Curtis C.W., Iwahori N., Kilmoyer R. *Hecke algebras and characters of parabolic type of finite groups with (B, N) -pairs*. Publ. Math. I.H.E.S., 40 (1971), pp. 81-116
- [2] Fulton, W. *Young tableaux: with applications to representation theory and geometry*. Cambridge University Press (1997)
- [3] Gorin, V., Kerov S., Vershik A. *Finite traces and representations of the group of infinite matrices over a finite field*. Preprint, arXiv:1209.4945
- [4] Humphreys, J.E. (1987), *The Steinberg representation*, Bull. Amer. Math. Soc. (N.S.) 16 (2): 237–263,
- [5] Neretin, Yu. A. *Lectures on Gaussian integral operators and classical groups*. EMS Series of Lectures in Mathematics. Europ. Math. Soc., Zürich, 2011.
- [6] Neretin, Yu.A. *The space L^2 on semi-infinite Grassmannian over finite field*, Adv. Math., 250 (2014), 320-350
- [7] Neretin, Yu.A. *On multiplication of double cosets for $\text{GL}(\infty)$ over a finite field*. Preprint, arXiv:1310.1596
- [8] R. Steinberg, *Collected Papers*, Amer. Math. Soc. (1997) pp. 580–586.
- [9] Vershik, A. M.; Kerov, S. V. *On an infinite-dimensional group over a finite field*, Funct. Anal. Appl. 32 (1998), no. 3, 3–10.
- [10] Vershik, A. M.; Kerov, S. V. *Four drafts on the representation theory of the group of infinite matrices over a finite field*. J. of Math. Sci. (New York), 2007, 147:6, 7129-7144

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